PHYSICAL REVIEW E **80**, 065204(R) (2009)

Partial time-delay coupling enlarges death island of coupled oscillators

Wei Zou[1,2] and Meng Zhan[1,*]

[1]*Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, China*

[2]*Graduate School of the Chinese Academy of Sciences, Beijing 100049, China*

(Received 11 October 2009; published 16 December 2009)

Coupling (or connection) in complex systems is of crucial importance in determining the system's dynamics and realizing certain system's functions. In this work, we propose a coupling form, partial time-delay coupling in coupled oscillator systems, in which some oscillators are time-delay coupled and the others remain instantaneously coupled, and study its impact on dynamics. We find that the partial time-delay coupling greatly enlarges the domain of oscillation death island in parameter space. In particular, the *smaller* the ratio p of time-delay coupled oscillators, the larger death island is. For a sufficiently large system, a universal amplification scaling, $R=p^{-1}(0<p\leq 1)$, is uncovered. These findings are proved to be very general and suggest that a tiny amount of time-delay couplings in coupled systems may tremendously change the dynamics of whole systems.

DOI: 10.1103/PhysRevE.80.065204 PACS number(s): 05.45.Xt, 87.10.−e

Emergent phenomena are omnipresent and spontaneously appear in ensembles of interacting individual elements [1–3]. Characterizing these collective behaviors and investigating transitions from one to another have aroused general interest among researchers due to their significance for both theoretic development and various applications. The model of coupled nonlinear equations provides a simple but powerful mathematical means for the study. In this work, we will propose a coupling form, partial time-delay coupling, and explore its peculiar effect on system's dynamics.

Researchers found that under certain conditions, coupled oscillatory (either periodic or chaotic) systems can cease oscillation and transit to steady states. The phenomenon is referred to as oscillation (or amplitude) death [4–6]. The steady states can either be those, which exist and are unstable in the uncoupled system, or be entirely newly created by the coupling. For the occurrence, at least one of the following conditions is required: (a) the parameters of coupled systems are sufficiently disparate [4–9]; (b) the interaction between coupling units includes a time delay [10–16]; (c) the coupling is dynamic [17,18]; (d) the coupled oscillators evolve with different time scales [19]; or (e) the oscillators are coupled via dissimilar (or conjugate) variables [20,21]. Obviously oscillation death plays a crucial role in many important dynamical processes of physical, chemical, and biological systems. For instance, recent studies have clearly shown that prior to the synchronization of cell-density-dependent glycolytic oscillators in yeast (or in large populations of chemical oscillators), desynchronization is well characterized by the disappearance of oscillations [22,23]. A study also revealed that the oscillation death appears in synthetic genetic networks, coexisting with other rich dynamical behaviors, which are significant for improving both adaptability and robustness of cellular population [24].

The time-delay-induced oscillation death was first reported by Reddy *et al.* in 1998 [10]. Since then, this phenomenon has been extensively observed in numerous systems, such as in a pair of electronic circuits [25], coupled living oscillators [26], thermo-optical oscillators [27], and laser systems [28]. Most of previous studies have been confined only to the homogeneous time-delay coupling, with which all oscillators are connected by an identical-time-delayed (or distributed-time-delayed [15]) signal. It is natural to ask what will happen in coupled oscillator systems with partial time-delay coupling, with which some oscillators are time-delay coupled and the others remain instantaneously coupled. Because time delay is ubiquitous in nature, arising due to finite propagation speeds of signals, the connection with a finite lag time can be approximated by a time-delay coupling, and the connection with a very fast propagation speed and consequently a very short time delay can be well approximated by an instantaneous coupling, such a partial time-delay coupling may naturally occur in realistic situations, as a bridge linking instantaneous coupling in the absence of any delays and full time-delay interaction. One may intuitively believe that partial time-delay coupling can only have a "partial" weak effect on the overall dynamics; namely, with increase in the number of time-delayed oscillators affected by the partial time-delay coupling, the system will be gradually influenced and become easier for the change in dynamics, such as the appearance of oscillation death. Our study in this work, however, gives a completely opposite result.

We start with the following two coupled Landau-Stuart oscillators:

$$\dot{z_j}(t) = (1 + iw - |z_j(t)|^2)z_j(t) + K[z_s(t-\tau_s) - z_j(t)] \quad (1)$$

for j, $s=1$ or 2, where z_1 and z_2 are complex, w is the natural frequency of single uncoupled oscillator, K is the coupling strength, and τ_1 and τ_2 are the time delays. Without coupling ($K=0$), each oscillator shows a stable limit cycle $z(t)=e^{iwt}$ accompanying with an unstable focus located at the origin ($z=0$). Under appropriate choices of K, τ_1, and τ_2, the origin becomes stable and oscillation death occurs. According to the analyses of previous studies [6,10], oscillation death is impossible for $\tau_1=\tau_2=0$ and large death islands can be found for $\tau_1=\tau_2=\tau\neq 0$ on the (τ,K) plane. For arbitrarily chosen τ_1

*Corresponding author; zhanmeng@wipm.ac.cn

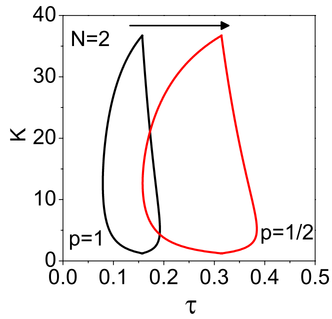

FIG. 1. (Color online) The oscillation death islands of coupled system (1) for the full time-delay coupling ($\tau_1=\tau_2=\tau$, $p=1$, the left one) and the partial time-delay coupling ($\tau_1=\tau$, $\tau_2=0$, $p=1/2$, the right one). The expansion direction is horizontal as illustrated by the arrow. $N=2$.

and τ_2, following the analysis of Reddy *et al.* [10], we obtain the following death boundary curves:

$$\frac{\tau_1+\tau_2}{2}=\frac{\cos^{-1}(1-1/K)}{w-\sqrt{2K-1}},$$

$$\frac{\tau_1+\tau_2}{2}=\frac{\pi-\cos^{-1}(1-1/K)}{w+\sqrt{2K-1}}. \qquad (2)$$

Obviously, the death regions are determined only by the average of the two delays τ_1 and τ_2, and we immediately find that the death island for $\tau_1=\tau$ and $\tau_2=0$ (the partial delay coupling between the two coupled oscillators) expands exactly twice along the τ direction as that for $\tau_1=\tau_2=\tau$ (the full delay coupling). The result is shown in Fig. 1 with $w=10$ chosen and fixed throughout the Rapid Communication, and the expansion direction is emphasized by a horizontal arrow. Similar result can be found for larger w, with which more than one death island is possible.

Next we consider the globally coupled N Landau-Stuart oscillators

$$\dot{z_j}(t)=(1+iw-|z_j(t)|^2)z_j(t)+\frac{2K}{N}\sum_{\substack{s=1\\ s\neq j}}^{N}[z_s(t-\tau_s)-z_j(t)] \qquad (3)$$

for $j=1,\ldots,N$. Equation (3) can degenerate to Eq. (1) for $N=2$, $\tau_1=\tau$, and $\tau_2=0$. For convenience, we introduce p as the ratio of time-delay coupled oscillators. Without losing generality, we set $\tau_s=\tau$ for $j=1,\ldots,pN$ and $\tau_s=0$ for $j=pN+1,\ldots,N$. Thus, the oscillators in the coupled system are divided into two parts based on their different time-delay relations with the rest: the oscillators from 1 to pN receive time-lagged information from all other oscillators, and oppositely the oscillators from $pN+1$ to N are instantaneously coupled with all others.

For one extreme case: the full time-delay coupling ($p=1$), the explicit result has already been obtained by Reddy *et al.* [10] with the oscillation death island boundaries described by

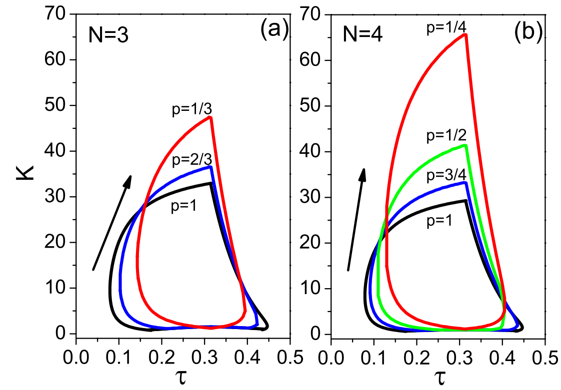

FIG. 2. (Color online) [(a) and (b)] The oscillation death islands of coupled system (3) for $N=3$ and 4, respectively. Now the expansion directions become sloped as shown by the two arrows.

$$\tau=\frac{\cos^{-1}q}{w-\sqrt{4Kb-1}},\qquad \tau=\frac{2\pi-\cos^{-1}q}{w+\sqrt{4Kb-1}},$$

$$\tau=\frac{\cos^{-1}\left(\frac{b}{b-1}q\right)}{w+\sqrt{r}},\qquad \tau=\frac{2\pi-\cos^{-1}\left(\frac{b}{b-1}q\right)}{w-\sqrt{r}}, \qquad (4)$$

where $q=1-1/(2Kb)$, $r=4K^2-1+4Kb(1-2K)$, and $b=1-1/N$. For $0<p<1$ (the partial time-delay coupling), however, no explicit forms of death boundaries are available and we have to rely on numerical simulations.

The numerical results for $N=3$ and $N=4$ are given in Figs. 2(a) and 2(b), respectively. To our surprise, the expansion of death island for smaller p occurs again, but the direction becomes sloped, as illustrated by the arrows in Fig. 2. In this respect, again the oscillation dynamics is greatly changed for smaller p.

This pattern keeps for a sufficiently large N, but the expansion for p from large to small becomes stretching exactly along the vertical direction on the (τ,K) parameter space. As an example, the oscillation death islands for different p's and fixed $N=200$ are illustrated in Fig. 3(a). A vertical arrow is added to guide the eyes. To quantify this size enlargement, we introduce a normalized amplification factor $R=S_p/S_{p=1}$, where S_p denotes the area of death island for any p and $S_{p=1}$ for $p=1$. Obviously, for the extreme case, $p=0$ (the instantaneous coupling), $R=0$, and for the other extreme case, $p=1$ (the full time-delay coupling), $R=1$. For $0<p<1$ (the partial time-delay coupling), our numerical experiments show a clear reciprocal relation between p and R, i.e.,

$$R=p^{-1}, \qquad (5)$$

which can be seen from Fig. 3(b), where the theoretical value (solid line) is in good agreement with the numerical result (squares) from Fig. 3(a). Clearly it is distinct from our intuitive idea for a monotonic increase relation of p and R. To emphasize the high jump of R from zero to a large value around $p=0$ and the slow damping value $R(R=1)$ at $p=1$, we use a circle and a triangle to show their differences. Note that the scaling law [Eq. (5)] is nearly unchanged if we increase

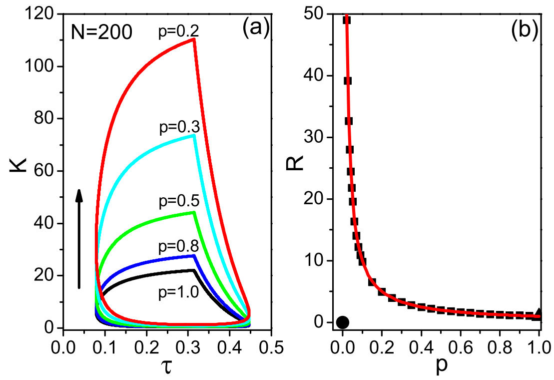

FIG. 3. (Color online) (a) The same as Fig. 2 for $N=200$ instead. The death islands expand greatly and exactly along the vertical direction. (b) The normalized amplification factor R is plotted with p, with the data obtained from (a) (square points) and the fit (solid line), $R=p^{-1}$. To emphasize the two extremes at $p=0$ $(R=0)$ and $p=1$ $(R=1)$, a circle and a triangle points are superimposed.

N further, and it is expected to be valid for any sufficiently large N. It is also notable that in numerical simulations, random initial conditions have been used, which signals that the time-delay-induced steady states within the oscillation death islands are not only locally but also globally stable.

Below we give a solid support for this scaling, by performing rigorous mathematical analysis. Assuming that coupled system (3) is divided into two groups, in which all elements are identical, and setting $z_j=A(j=1,\ldots,pN)$ for the time-delay coupled elements and $z_j=B(j=pN+1,\ldots,N)$ for the others, we have

$$\dot{A}(t)=[1+iw-|A(t)|^2]A(t)+2K\left(p-\frac{1}{N}\right)[A(t-\tau)-A(t)]+2K(1-p)[B(t-\tau)-A(t)],$$

$$\dot{B}(t)=[1+iw-|B(t)|^2]B(t)+2Kp[A(t)-B(t)]. \quad (6)$$

Then the reduced system can be easily treated. Similar treatment can be found in Ref. [29] for the analysis of age phenomenon. Its characteristic equation about the origin $(A=B=0)$ is given by

$$\lambda^2-\alpha\lambda+\beta=0, \quad (7)$$

with

$$\alpha=2\left[1-K-Kp+\frac{K}{N}+K\left(p-\frac{1}{N}\right)e^{-\lambda\tau}\pm iw\right],$$

$$\beta=\left(1-2K+\frac{2K}{N}\pm iw\right)(1-2kp\pm iw)+2Ke^{-\lambda\tau}\left[p\left(1-2K+\frac{2K}{N}\pm iw\right)-\frac{1}{N}(1\pm iw)\right].$$

For an infinitely large $N(N\to\infty)$, the eigenvalues can be explicitly expressed as

$$\lambda=1-2K\pm iw,$$

$$\lambda=1-2Kp\pm iw+2Kpe^{-\lambda\tau}. \quad (8)$$

Therefore, the critical curves of oscillation death island are determined by

$$K\geq\frac{1}{2},$$

$$\tau=\frac{\cos^{-1}[1-1/(2Kp)]}{w-\sqrt{4Kp-1}},$$

$$\tau=\frac{2\pi-\cos^{-1}[1-1/(2Kp)]}{w+\sqrt{4Kp-1}}. \quad (9)$$

As a result, the combined term of Kp in the above equations clearly demonstrates our numerical observations in Fig. 3 for both the vertical expansion direction and the magnification scaling in Eq. (5).

More interestingly, all of the above results are not restricted to the specific oscillator model, the Landau-Stuart oscillator in Eq. (1), and they hold fairly generally irrespective of the nature of single oscillator. Consider the general form of globally coupled oscillators

$$\dot{X}_j=F(X_j)+\frac{2K}{N}\sum_{\substack{s=1\\ s\neq j}}^{N}\Gamma[X_s(t-\tau_s)-X_j(t)] \quad (10)$$

for $j=1,\ldots,N$, where $X_j\in R^n$ is the state vector of the jth element and Γ is a $n\times n$ constant linking matrix. Again, the ratio of the time-delay coupled oscillators is set to be p. The dynamical system $\dot{X}=F(X)$ is generally capable of exhibiting nonstationary (periodic or even chaotic) behavior coexisting with an unstable focus X^*. Denoting a pair of leading conjugate characteristic eigenvalues associated with the unstable X^* are $\lambda_R\pm i\lambda_I$ ($\lambda_R>0$ and $\lambda_I>0$), and assuming a unit matrix Γ, it is not difficult to get the following analytic results.

(i) If $p=0$, the oscillation death state $X_j=X^*$ is always unstable. Thus, we can never find death in the coupled identical oscillators without time-delay interactions.

(ii) For $N=2$, the death island boundaries are given by

$$\frac{\tau_1+\tau_2}{2}=\frac{\cos^{-1}(1-\lambda_R/K)}{\lambda_I-\sqrt{2K\lambda_R-\lambda_R^2}},$$

$$\frac{\tau_1+\tau_2}{2}=\frac{\pi-\cos^{-1}(1-\lambda_R/K)}{\lambda_I+\sqrt{2K\lambda_R-\lambda_R^2}}. \quad (11)$$

Hence the feature of τ-direction expansion two times from the full time-delay coupling $(p=1)$ to the partial time-delay coupling $(p=1/2)$ is unchanged.

(iii) For $p=1$ $(N\geq3)$, the death island boundaries are

$$\tau=\frac{\cos^{-1}q}{\lambda_I-\sqrt{4Kb\lambda_R-\lambda_R^2}},\quad \tau=\frac{2\pi-\cos^{-1}q}{\lambda_I+\sqrt{4Kb\lambda_R-\lambda_R^2}},$$

$$\tau = \frac{\cos^{-1}\left(\frac{b}{b-1}q\right)}{\lambda_I + \sqrt{r}}, \quad \tau = \frac{2\pi - \cos^{-1}\left(\frac{b}{b-1}q\right)}{\lambda_I - \sqrt{r}}, \quad (12)$$

where $q=1-\lambda_R/(2Kb)$, $r=4K^2-\lambda_R^2+4Kb(\lambda_R-2K)$, and $b=1-1/N$.

(iv) For an infinite system ($N\to\infty$), the death island boundaries are determined by

$$K \geq \frac{\lambda_R}{2},$$

$$\tau = \frac{\cos^{-1}[1-\lambda_R/(2Kp)]}{\lambda_I - \sqrt{4Kp\lambda_R - \lambda_R^2}},$$

$$\tau = \frac{2\pi - \cos^{-1}[1-\lambda_R/(2Kp)]}{\lambda_I + \sqrt{4Kp\lambda_R - \lambda_R^2}}, \quad (13)$$

which clearly reveals that partial time-delay coupling enlarges the death island p^{-1} times in the coupling strength K direction compared with that of the full time-delay coupling ($p=1$).

All of the analytical results have been well tested by our numerical simulations with several specific oscillator models. Quite similar phenomena are observed even for nonunit linking matrix Γ. The same as the global coupling studied in the Rapid Communication, for the nearest-neighbor diffusive coupling, we find that substitution of even one connection of instantaneous coupling with time-delay coupling indeed enlarges the death island tremendously. The remarkable finding of enlargement for general dynamical system (10) and any oscillator number N (Figs. 1–3) could shed an improved light on our understanding of (partial) time-delay coupling in coupled systems. As the instantaneous coupling ($p=0$) and the full time-delay coupling ($p=1$) are two extreme cases keeping their full symmetries, and all partial time-delay couplings ($0<p<1$) correspond to intermediate cases, it is reasonable to understand that the introduction of just a single time-delayed oscillator into the system with the most asymmetrical structure can have the most dramatic effect. The oscillation elimination by the input of a tiny amount of time-delay couplings (particularly, even one time-delay connection in some cases) may be properly used in dynamical control with a strikingly high efficiency and may have already been exploited by nature such as cellular population for the variation in certain biological functions [22–24].

This work was supported by the Outstanding Oversea Scholar Foundation of Chinese Academy of Sciences (Bairenjihua).

[1] A. T. Winfree, *The Geometry of Biological Time* (Springer-Verlag, New York, 1980).
[2] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).
[3] S. H. Strogatz, *Sync: The Emerging Science of Spontaneous Order* (Hyperion, New York, 2003).
[4] K. Bar-Eli, Physica D **14**, 242 (1985).
[5] R. E. Mirollo and S. H. Stogatz, J. Stat. Phys. **60**, 245 (1990).
[6] D. G. Aronson, G. B. Ermentrout, and N. Kopell, Physica D **41**, 403 (1990).
[7] J. Yang, Phys. Rev. E **76**, 016204 (2007).
[8] A. Stefanski and T. Kapitaniak, Phys. Lett. A **210**, 279 (1996).
[9] G. V. Osipov, A. S. Pikovsky, M. G. Rosenblum, and J. Kurths, Phys. Rev. E **55**, 2353 (1997).
[10] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Phys. Rev. Lett. **80**, 5109 (1998).
[11] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Physica D **129**, 15 (1999).
[12] R. Dodla, A. Sen, and G. L. Johnston, Phys. Rev. E **69**, 056217 (2004).
[13] A. Prasad, Phys. Rev. E **72**, 056204 (2005).
[14] F. M. Atay, Physica D **183**, 1 (2003).
[15] F. M. Atay, Phys. Rev. Lett. **91**, 094101 (2003).
[16] K. Konishi, Phys. Rev. E **70**, 066201 (2004).
[17] K. Konishi, Phys. Rev. E **68**, 067202 (2003).
[18] K. Konishi, Int. J. Bifurcation Chaos Appl. Sci. Eng. **17**, 2781 (2007).
[19] S. D. De Monte, F. d'Ovidio, and E. Mosekilde, Phys. Rev. Lett. **90**, 054102 (2003).
[20] R. Karnatak, R. Ramaswamy, and A. Prasad, Phys. Rev. E **76**, 035201(R) (2007).
[21] W. Zou and M. Zhan, Fron. Phys. China **4**, 97 (2009).
[22] S. De Monte, F. d'Ovidio, S. Danø, and P. G. Sørensen, Proc. Natl. Acad. Sci. U.S.A. **104**, 18377 (2007).
[23] A. F. Taylor, M. R. Tinsley, F. Wang, Z. Y. Huang, and K. Showalter, Science **323**, 614 (2009).
[24] E. Ullner, A. Zaikin, E. I. Volkov, and J. García-Ojalvo, Phys. Rev. Lett. **99**, 148103 (2007).
[25] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Phys. Rev. Lett. **85**, 3381 (2000).
[26] A. Takamatsu, T. Fujii, and I. Endo, Phys. Rev. Lett. **85**, 2026 (2000).
[27] R. Herrero, M. Figueras, J. Rius, F. Pi, and G. Orriols, Phys. Rev. Lett. **84**, 5312 (2000).
[28] A. Prasad, Y. C. Lai, A. Gavrielides, and V. Kovanis, Phys. Lett. A **318**, 71 (2003).
[29] H. Daido and K. Nakanishi, Phys. Rev. Lett. **93**, 104101 (2004).